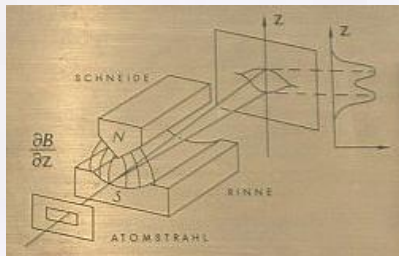


Spin Eigenstates - Review

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SG Devices Measure Spin

- ▶ Orient device in direction \mathbf{n}
- ▶ The representation of $|\psi\rangle$ in the S_n -basis for spin $\frac{1}{2}$:

$|\psi\rangle_n = I_n|\psi\rangle$, where

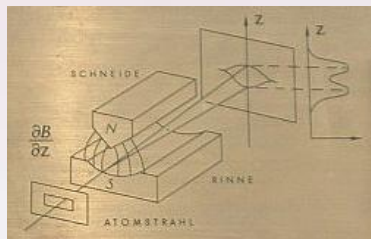
$$I_n = |+\mathbf{n}\rangle\langle+\mathbf{n}| + |-\mathbf{n}\rangle\langle-\mathbf{n}|$$

$$|\psi\rangle_n = |+\mathbf{n}\rangle\langle+\mathbf{n}|\psi\rangle + |-\mathbf{n}\rangle\langle-\mathbf{n}|\psi\rangle$$

$$= a_+|+\mathbf{n}\rangle + a_-|-\mathbf{n}\rangle$$

$$\rightarrow \begin{pmatrix} \langle+\mathbf{n}|\psi\rangle \\ \langle-\mathbf{n}|\psi\rangle \end{pmatrix}$$

- ▶ $\text{Prob}(|+\mathbf{n}\rangle) = |\langle+\mathbf{n}|\psi\rangle|^2$



Representation of Operators

Matrix Representation of \hat{A} in S_n -basis

$$\hat{A} \rightarrow A_n = \begin{pmatrix} \langle +n|\hat{A}|+n\rangle & \langle +n|\hat{A}|-n\rangle \\ \langle -n|\hat{A}|+n\rangle & \langle -n|\hat{A}|-n\rangle \end{pmatrix}$$

Matrix Representations

$$\hat{A} \rightarrow A_n = S^\dagger A_z S,$$

where

$$S = \begin{pmatrix} \langle +z|+n\rangle & \langle +z|-n\rangle \\ \langle -z|+n\rangle & \langle -z|-n\rangle \end{pmatrix}$$

and

$$A_z = \begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}$$

Change of Basis (z to n) - States

Transform Kets

$$\begin{aligned} |\psi\rangle_z &= |+\!z\rangle\langle+\!z|\psi\rangle + |-\!z\rangle\langle-\!z|\psi\rangle \\ &= (|+\!n\rangle\langle+\!n| + |-\!n\rangle\langle-\!n|) |+\!z\rangle\langle+\!z|\psi\rangle \\ &\quad + (|+\!n\rangle\langle+\!n| + |-\!n\rangle\langle-\!n|) |-\!z\rangle\langle-\!z|\psi\rangle \\ &= [\langle+\!n|+\!z\rangle\langle+\!z|\psi\rangle + \langle+\!n|-\!z\rangle\langle-\!z|\psi\rangle] |+\!n\rangle \\ &\quad + [\langle-\!n|+\!z\rangle\langle+\!z|\psi\rangle + \langle-\!n|-\!z\rangle\langle-\!z|\psi\rangle] |-\!n\rangle \end{aligned}$$

Matrix Representation $|\psi\rangle_n = \hat{S}^\dagger |\psi\rangle_z$

$$\begin{aligned} \begin{pmatrix} \langle+\!n|\psi\rangle \\ \langle-\!n|\psi\rangle \end{pmatrix} &= \underbrace{\begin{pmatrix} \langle+\!n|+\!z\rangle & \langle+\!n|-\!z\rangle \\ \langle-\!n|+\!z\rangle & \langle-\!n|-\!z\rangle \end{pmatrix}}_{\text{Components of } z \text{ states}} \begin{pmatrix} \langle+\!z|\psi\rangle \\ \langle-\!z|\psi\rangle \end{pmatrix} \\ &\equiv S^\dagger \begin{pmatrix} \langle+\!z|\psi\rangle \\ \langle-\!z|\psi\rangle \end{pmatrix} \end{aligned}$$

Change of Basis (z to n) - Operators

Begin with States

$$|\psi\rangle_n = \hat{S}^\dagger |\psi\rangle_z, \quad {}_n\langle\psi| = {}_z\langle\psi| \hat{S},$$

where

$$S^\dagger = \begin{pmatrix} \langle +n|+z\rangle & \langle +n|-z\rangle \\ \langle -n|+z\rangle & \langle -n|-z\rangle \end{pmatrix}$$

Relate $\langle\psi|\hat{A}|\psi\rangle$ in z-basis to value in n-basis, using $\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = I$.

$$\begin{aligned} {}_z\langle\psi|\hat{A}_z|\psi\rangle_z &= {}_z\langle\psi|\hat{S}\hat{S}^\dagger\hat{A}_z\hat{S}\hat{S}^\dagger|\psi\rangle_z \\ &= {}_n\langle\psi|\hat{S}^\dagger\hat{A}_z\hat{S}|\psi\rangle_n \\ &= {}_n\langle\psi|\hat{A}_n|\psi\rangle_n \end{aligned} \tag{1}$$

If we define $\hat{A}_{\text{new}} = \hat{A}_n$ and $\hat{A}_{\text{old}} = \hat{A}_z$, then

$$\hat{A}_{\text{new}} = \hat{S}^\dagger \hat{A}_{\text{old}} \hat{S}$$

Angular Momentum Operators

Rotations and Generators (\hat{J}_n is Hermitian)

$$\hat{R}(\phi \mathbf{n}) = e^{-i\hat{J}_n \phi / \hbar}$$

Commutation Relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y.$$

Operators

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \quad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

Eigenstates $(2j + 1)$ for $-j, -j + 1, \dots, j - 1, j$

$$\hat{J}^2 |j, m\rangle = j(j + 1)\hbar^2 |j, m\rangle$$

$$\hat{J}_z |j, m\rangle = m\hbar |j, m\rangle$$

$$\hat{J}_\pm |j, m\rangle = \sqrt{j(j + 1) - m(m \pm 1)}\hbar |j, m \pm 1\rangle$$

Spin $\frac{1}{2}$ Representations

- ▶ Operators $\hat{S}^2, \hat{S}_z, \hat{S}_\pm$
- ▶ States $|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle,$
- ▶ Representations $\hat{\mathbf{S}} \rightarrow \frac{\hbar}{2}\boldsymbol{\sigma}$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- ▶ Expectation Values

$$\langle S_x \rangle = \langle \psi | \hat{S}_x | \psi \rangle = \begin{pmatrix} \langle +z | \psi \rangle^* & \langle -z | \psi \rangle^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix}$$

Spin 1 Representations

- ▶ Operators $\hat{S}^2, \hat{S}_z, \hat{S}_\pm$
- ▶ States $|1, 1\rangle, |1, 0\rangle, |1, -1\rangle,$
- ▶ Representations

$$\hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \hat{S}_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

- ▶ Representation of \hat{A} :

$$\hat{A} \rightarrow \begin{pmatrix} \langle 1, 1 | \hat{A} | 1, 1 \rangle & \langle 1, 1 | \hat{A} | 1, 0 \rangle & \langle 1, 1 | \hat{A} | 1, -1 \rangle \\ \langle 1, 0 | \hat{A} | 1, 1 \rangle & \langle 1, 0 | \hat{A} | 1, 0 \rangle & \langle 1, 0 | \hat{A} | 1, -1 \rangle \\ \langle 1, -1 | \hat{A} | 1, 1 \rangle & \langle 1, -1 | \hat{A} | 1, 0 \rangle & \langle 1, -1 | \hat{A} | 1, -1 \rangle \end{pmatrix}$$

Representation of \hat{S}_x - Spin 1 Case

Noting that $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$, $A_{ii} = 0$, $i = 1, 2, 3$, and

$$\begin{aligned}A_{12} &= \langle 1, 1 | \hat{S}_x | 1, 0 \rangle = A_{21}^* \\ &= \frac{1}{2} \langle 1, 1 | \hat{S}_+ + \hat{S}_- | 1, 0 \rangle \\ &= \frac{1}{2} \left(\sqrt{2} \langle 1, 1 | 1, 1 \rangle + \sqrt{2} \langle 1, 1 | 1, -1 \rangle \right) = \frac{1}{\sqrt{2}}.\end{aligned}$$

$$\begin{aligned}A_{13} &= \langle 1, 1 | \hat{S}_x | 1, -1 \rangle = A_{31}^* \\ &= \frac{1}{2} \langle 1, 1 | \hat{S}_+ + \hat{S}_- | 1, -1 \rangle = 0.\end{aligned}$$

$$\begin{aligned}A_{23} &= \langle 1, 0 | \hat{S}_x | 1, -1 \rangle = A_{32}^* \\ &= \frac{1}{2} \langle 1, 0 | \hat{S}_+ + \hat{S}_- | 1, -1 \rangle = \frac{1}{\sqrt{2}}.\end{aligned}$$

The final representation is ...

Eigenstates of \hat{S}_x - Spin 1 Case

Find the eigenstates of S_x in S_z -basis

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Eigenvalues [$\lambda = -1, 0, 1$]

$$\begin{vmatrix} -\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda \left(\lambda^2 - \frac{1}{2} \right) + \frac{1}{2} \lambda = 0$$

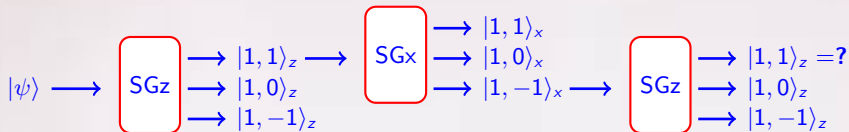
Eigenvectors for $\lambda = -1$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$b = -\sqrt{2}a = -\sqrt{2}c, a + c = -\sqrt{2}b, \Rightarrow |1, -1\rangle_x \xrightarrow{S_z} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Spin 1 Particles - SG Devices

Send spin 1 particles through 3 Stern Gerlach devices.



Probability [to find $|\psi\rangle$ in state $|\phi\rangle$] = $|\langle\phi|\psi\rangle|^2$,

$$|_z\langle 1, 1|\psi\rangle|^2, \quad |_x\langle 1, -1|1, 1\rangle_z|^2, \quad |_z\langle 1, 1|1, -1\rangle_x|^2,$$

Example: Evaluate $|_x\langle 1, -1|1, 1\rangle_z|^2$.

Recall that $|1, -1\rangle_x$ is an eigenstate of S_x :

$$\text{Therefore, } |1, -1\rangle_x \xrightarrow{S_z} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow |_x\langle 1, -1|1, 1\rangle_z|^2 = \frac{1}{4}.$$

Time Evolution

The Schrödinger Equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

For \hat{H} time-independent,

$$|\psi(t)\rangle = \hat{U} |\psi(0)\rangle, \quad \hat{U} = e^{-i\hat{H}t/\hbar}$$

Energy Eigenstates $\hat{H}|E\rangle = E|E\rangle$,

Time Evolution: Initial State $|\psi(0)\rangle = \sum_n |E_n\rangle \langle E_n | \psi(0)\rangle$

$$\Rightarrow |\psi(t)\rangle = \sum_n e^{-i\hat{E}_n t/\hbar} |E_n\rangle \langle E_n | \psi(0)\rangle$$

Expectation Values: $i\hbar \frac{d}{dt} \langle A \rangle = \langle \psi(t) | [\hat{H}, \hat{A}] | \psi(t)\rangle + \frac{\partial \hat{A}}{\partial t} | \psi(t)\rangle$

Precession in a Magnetic Field

Constant Magnetic Field $\mathbf{B} = B_0 \mathbf{k}$

$$\hat{H} = -\hat{\mu} \cdot \mathbf{B} = \frac{ge}{2mc} B_0 S_z \equiv \omega_0 S_z$$

Evolution of States $\hat{H}|\pm \mathbf{z}\rangle = \pm \frac{\hbar\omega_0}{2}|\pm \mathbf{z}\rangle$

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\hat{H}t/\hbar} |+\mathbf{x}\rangle \\ &= \frac{1}{\sqrt{2}} \left(e^{-iE_+t/\hbar} |+\mathbf{z}\rangle + e^{-iE_-t/\hbar} |-\mathbf{z}\rangle \right) \\ &= \frac{1}{\sqrt{2}} \left(e^{-i\omega_0 t/2} |+\mathbf{z}\rangle + e^{i\omega_0 t/2} |-\mathbf{z}\rangle \right) \end{aligned}$$

Expectation values

$$\langle S_z \rangle = 0, \quad \langle S_x \rangle = \frac{\hbar}{2} \cos \omega_0 t, \quad \langle S_y \rangle = \frac{\hbar}{2} \sin \omega_0 t$$

Uncertainty

For Hermitian matrices \hat{A} , \hat{B} , $[\hat{A}, \hat{B}] = i\hat{C}$,

$$\Delta A \Delta B \geq \frac{|\langle C \rangle|}{2}$$

Example: $\Delta J_x \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle|$

Recall

$$\Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2$$

and

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Two Spin Systems

Basis States for Spin- $\frac{1}{2}$ Particles

$$|+z, +z\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$|+x, +z\rangle = \frac{1}{\sqrt{2}}|+z, +z\rangle + \frac{1}{\sqrt{2}}|-z, +z\rangle$$

Hyperfine Splitting $\hat{H} = \frac{2A}{\hbar^2} \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$

$$\hat{H} = \frac{A}{\hbar^2} (\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z})$$

$$\rightarrow \begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} & A & 0 \\ 0 & A & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix} \quad (2)$$

Eigenvalue Problem

Seek Eigenvalues and Energy Eigenstates $\hat{H}|E\rangle = E|E\rangle$.

$$\begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} & A & 0 \\ 0 & A & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = E \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Eigenvalue Equation

$$\begin{aligned} 0 &= \begin{vmatrix} \frac{A}{2} - E & 0 & 0 & 0 \\ 0 & -\frac{A}{2} - E & A & 0 \\ 0 & A & -\frac{A}{2} - E & 0 \\ 0 & 0 & 0 & \frac{A}{2} - E \end{vmatrix} \\ &= \left(\frac{A}{2} - E\right)^2 \left[\left(E + \frac{A}{2}\right)^2 - A^2 \right] \end{aligned}$$

Energy Eigenstates

For eigenvalues $E = \frac{A}{2}$, we get the triplet

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{or}$$

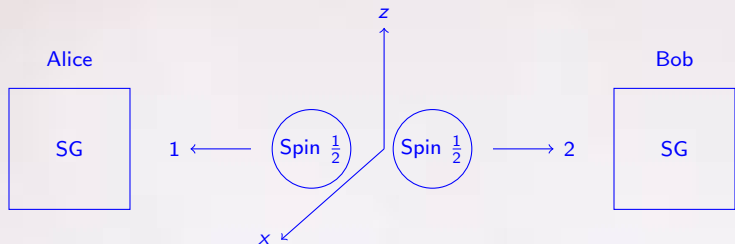
$$|1, 1\rangle = |+z, +z\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}}|+z, -z\rangle + \frac{1}{\sqrt{2}}|-z, +z\rangle,$$

$$|1, -1\rangle = |-z, -z\rangle.$$

For eigenvalues $E = -\frac{3A}{2}$, we get the singlet

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{or } |0, 0\rangle = \frac{1}{\sqrt{2}}|+z, -z\rangle - \frac{1}{\sqrt{2}}|-z, +z\rangle.$$

EPR Paradox - $|0, 0\rangle$ Decay



A spin-0 particle decays into two spin- $\frac{1}{2}$ particles.

$$\begin{aligned}|0, 0\rangle &= \frac{1}{\sqrt{2}}|+z, -z\rangle - \frac{1}{\sqrt{2}}|-z, +z\rangle \\ &= \frac{1}{\sqrt{2}}|+z\rangle_1|-z\rangle_2 - \frac{1}{\sqrt{2}}|-z\rangle_1|+z\rangle_2.\end{aligned}$$

What do Alice and Bob measure?